

# Superfluidity of fermion atoms loaded in a deep optical lattice: the existence of two rotonlike modes

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We present theoretical calculations of collective modes of the one-band attractive Hubbard model which is widely used to study the s-wave superfluid phases of atomic Fermi gases of two-hyperfine states loaded in a deep optical lattice. To make our theory applicable for both superconductivity and superfluidity, we assume the more general  $t-U-J$  Hamiltonian. Using the functional differentiation we derive Schwinger-Dyson equations for the single-particle Green's functions. The method of Legendre transform is used to give a systematic derivation of the Bethe-Salpeter (BS) equation for the two-particle Green's function and the associated collective modes. The numerical solution of the BS equation in the limit  $J \rightarrow 0$  shows the existence of two rotonlike collective modes with different low-energy Goldstone dispersions and different positions of the rotonlike minima. The two rotonlike modes lie outside of the region determined by the lower boundary of the particle-hole continuum, and therefore, the two modes are not damped and they should be experimentally observable. In the presence of superfluid flow at a certain critical flow momentum, the minimum of the first rotonlike mode reaches zero energy, but this occurs before the minimum of the second mode and the lower boundary of the particle-hole continuum do, i.e. there are two critical flow momenta related to the existence of two rotonlike excitations.

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## I. INTRODUCTION

### A. The Hamiltonian

One of the key questions of condensed matter physics is to understand the nature of single particle and collective excitations in high-temperature superconductors. It is widely accepted that the Hubbard model with an attractive on-site interaction plays an important role in the qualitative understanding of s-wave superconductors, but there is no consensus, however, about the theoretical models needed to understand the high-temperature superconductivity. The  $t-U-J$  model with a repulsive on-site interaction can serve as a possible model to study high-temperature superconductivity. This model was first used in connection with gossamer superconductivity,<sup>1,2</sup> but it also describes the opening of a d-wave pairing gap in cuprite compounds, and is consistent with the fact that the basic pairing mechanism arises from the antiferromagnetic (AF) exchange correlations. On the experimental side one encounters the problem of changing the system parameters  $t, U$  and  $J$  for a given superconductor, because different parameter sets require different types of superconducting material.

Ultracold atomic Fermi gases loaded in optical lattices present a new opportunity to overcome this obstacle, and to emulate high-temperature superconductors. Optical lattices realize the Hubbard model, if the lattice potential is sufficiently deep such that the tight-binding approximation is valid. In such a system the atom-atom interaction can be manipulated in a controllable way by changing the scattering length from the Bardeen-Cooper-Schrieffer (BCS) side (negative values) to the Bose-Einstein condensation (BEC) side (positive values), reaching very large values close to the Fesh-

bach resonance. On the BEC side of the resonance the pseudospin-up and pseudospin-down atoms can form diatomic molecules, and these bosonic molecules can undergo BEC at a low enough temperature.<sup>3</sup>

In what follows, we shall focus our attention on the BCS side where the existence of a superfluid phase of Fermi atoms is expected analogous to superconductivity. We shall examine the spectrum of the collective excitations of population-balanced atomic Fermi gases of two hyperfine states with contact interaction, loaded in optical lattices. The two hyperfine states are described by pseudospins. There are  $M$  atoms distributed along  $N$  sites, and the corresponding filling factor  $f = M/N$  is assumed to be smaller than unity. For a sufficiently deep lattice potential, the system is well described by the single-band attractive Hubbard model.

To make our theory applicable for both superconductivity and superfluidity, the attractive Hubbard model will be treated as a  $J \rightarrow 0$  limit of the more general  $t-U-J$  model defined by the following Hamiltonian:

$$H = - \sum_{i,j,\sigma} t_{ij} \psi_{i,\sigma}^\dagger \psi_{j,\sigma} + U \sum_i \hat{n}_{i,\uparrow} \hat{n}_{i,\downarrow} - \mu \sum_{i,\sigma} \hat{n}_{i,\sigma} + J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j \quad (1)$$

where  $t_{ij}$  is the single-electron hopping integral,  $\mu$  is the chemical potential, and  $\hat{n}_{i,\sigma} = \psi_{i,\sigma}^\dagger \psi_{i,\sigma}$  is the density operator on site  $i$ . The Fermi operator  $\psi_{i,\sigma}^\dagger$  ( $\psi_{i,\sigma}$ ) creates (destroys) a fermion on the lattice site  $i$  with pseudospin projection  $\sigma = \uparrow, \downarrow$ . The symbol  $\sum_{\langle i,j \rangle}$  means sum over nearest-neighbor sites of the two-dimensional lattice. The spin operator is defined by  $\vec{S}_i = (S_i^x, S_i^y, S_i^z) = \psi_{i,\sigma}^\dagger \vec{\sigma}_{\sigma\sigma'} \psi_{i,\sigma'}/2$ , where  $\vec{\sigma}$  is a vector formed by the

Pauli spin matrices  $(\sigma_x, \sigma_y, \sigma_z)$ . The AF interaction can be written as  $J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j = J \sum_{\langle i,j \rangle} S_i^z S_j^z + \frac{1}{2} J \sum_{\langle i,j \rangle} [S_i^+ S_j^- + S_i^- S_j^+]$ , where  $S_i^\pm = S_i^x \pm i S_i^y$ .

### B. Collective excitations in moving optical lattices in the GRPA

For the case when the periodic array of microtraps is generated by counter propagating laser beams with differing frequencies the optical lattice potential is moving with a velocity  $-\mathbf{v}$  (in the laboratory frame) with magnitude proportional to the relative frequency detuning of the two laser beams. In a frame fixed with respect to the lattice potential, the fermion atoms flow with a constant quasimomentum  $\mathbf{p} = m\mathbf{v}$ , where  $m$  is the mass of the loaded atoms. For population-balanced Fermi gases the order parameter field,  $\Phi_j(u) = -|U| \langle \psi_{j,\downarrow}(u) \psi_{j,\uparrow}(u) \rangle$  (or  $\Phi_j^*(u) = -|U| \langle \psi_{j,\uparrow}^\dagger(u) \psi_{j,\downarrow}^\dagger(u) \rangle$ ), in the mean-field approximation varies as  $\Phi_j \propto \Delta \exp[2i\mathbf{p} \cdot \mathbf{r}_j]$ . Here, the symbol  $\langle \rangle$  means ensemble average, and  $\Delta$  is a real quantity which depends on the lattice velocity.<sup>4,5</sup> In a moving lattice, the formation of BCS superfluidity is possible; but due to the presence of quasimomentum  $\mathbf{p}$ , the superflow can break down.

The stability of balanced superfluid Fermi gases loaded into a moving optical lattice has been recently studied using the second-order time-dependent perturbation theory,<sup>4</sup> and the Green's function formalism.<sup>5</sup> It was pointed out that the superfluid state could be destabilized at a critical flow momentum via two different mechanisms: depairing (pair-breaking) at  $\mathbf{p}_{pb}$  and Landau instabilities at  $\mathbf{p}_{cr}$ . The depairing takes place when the single fermionic excitations are broken, while the Landau instability is related to the rotonlike structure of the spectrum of the collective excitations. The superfluid state becomes unstable when the energy of the rotonlike minimum reaches zero at a given quasimomentum. The numerical solution of the number, gap and collective-mode equations shows that at a zero temperature the Landau instability appears before the depairing mechanism.<sup>5</sup>

### C. The Hubbard-Stratonovich transformation and the Bethe-Salpeter equation

It is known that the single-particle excitations of the Hamiltonian (1) manifest themselves as poles of the single-particle Green's function,  $G$ ; while the two-particle (collective) excitations could be related to the poles of the two-particle Green's function,  $K$ . The poles of these Green's functions are defined by the solutions of the Schwinger-Dyson (SD) equation  $G^{-1} = G^{(0)-1} - \Sigma$ , and the Bethe-Salpeter (BS) equation  $[K^{(0)-1} - I]\Psi = 0$ . Here,  $G^{(0)}$  is the free single-particle propagator,  $\Sigma$  is the electron self-energy,  $I$  is the BS kernel, and the two-particle free propagator  $K^{(0)} = GG$  is a product of two

fully dressed single-particle Green's functions. Since the electron self-energy depends on the two-particle Green's function, the positions of the poles have to be obtained by solving the SD and BS equations self-consistently.

It is widely accepted that the general random phase approximation (GRPA) is a good approximation for the collective excitations in a weak-coupling regime, and therefore, it can be used to separate the solutions of the SD and the BS equations. In this approximation, the single-particle excitations are replaced with those obtained by diagonalizing the Hartree-Fock (HF) Hamiltonian; while the collective modes are obtained by solving the BS equation in which the single-particle Green's functions are calculated in HF approximation, and the BS kernel is obtained by summing ladder and bubble diagrams.

Generally speaking, there exist two different GRPA that can be used to calculate the spectrum of the collective excitations of the Hubbard Hamiltonian in a stationary (or moving) optical lattice. The first approach uses the Green's function method,<sup>5-12</sup> while the second one is based on the Anderson-Rickayzen equations.<sup>4,13-15</sup>

The Green's function approach has been used to obtain the collective excitations in the problem of the exciton BEC<sup>6-8</sup> and in s-wave layered superconductors.<sup>9</sup> According to the Green's function method, the collective modes manifest themselves as poles of both the two-particle Green's function,  $K$ , and the density and spin response functions. The two response functions can be expressed in terms of  $K$ , but it is very common to obtain the poles of the density response function by following the Baym and Kadanoff formalism,<sup>16</sup> which uses functional derivatives of the density with respect to the external fields.<sup>5</sup>

The second method that can be used to obtain the collective excitation spectrum of the Hubbard Hamiltonian starts from the Anderson-Rickayzen equations, which in the GRPA can be reduced to a set of three coupled equations and the collective-mode spectrum is obtained by solving a  $3 \times 3$  secular determinant. Recently,<sup>4</sup> the Belkhir and Randeria  $3 \times 3$  secular determinant<sup>15</sup> has been generalized to the case of a moving optical lattice.

Instead of mean-field decoupling the quartic interaction terms, we apply the idea that we can transform these quartic terms into quadratic form by making the Hubbard-Stratonovich transformation for the electron operators. The attempt to decouple the quartic  $U$  and  $J$  terms by using two-component Nambu field operators  $\hat{\psi}(x) = \begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow^\dagger(x) \end{pmatrix}$  and  $\hat{\bar{\psi}}(y) = \begin{pmatrix} \psi_\uparrow^\dagger(y) & \psi_\downarrow(y) \end{pmatrix}$  does not work because the existence of  $S^\pm$  parts in the  $J$  interaction requires the use of more complicated four-component fermion field operators ( $\hat{\psi}$  and  $\hat{\bar{\psi}}$  obey anticommutation

relations):

$$\begin{aligned}\hat{\psi}(x) &= \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_{\uparrow}(x) \\ \psi_{\downarrow}(x) \\ \psi_{\uparrow}^{\dagger}(x) \\ \psi_{\downarrow}^{\dagger}(x) \end{pmatrix}, \\ \hat{\psi}(y) &= \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_{\uparrow}^{\dagger}(y) \psi_{\downarrow}^{\dagger}(y) \psi_{\uparrow}(y) \psi_{\downarrow}(y) \end{pmatrix}.\end{aligned}\quad (2)$$

In contrast to the previous approaches, such that after performing the Hubbard-Stratonovich transformation the fermion degrees of freedom are integrated out; we decouple the quartic problem by introducing a model system which consists of a four-component boson field  $A_{\alpha}(z)$  ( $\alpha = 1, 2, 3, 4$ ) interacting with fermion fields (2).

There are three advantages of keeping both the fermion and the boson degrees of freedom. First, the approximation that is used to decouple the self-consistent relation between the electron self-energy and the two-particle Green's function automatically leads to conserving approximations because it relies on the fact that the BS kernel can be written as functional derivatives of the Fock  $\Sigma^F$  and the Hartree  $\Sigma^H$  self-energy  $I = I_d + I_{exc} = \delta\Sigma^F/\delta G + \delta\Sigma^H/\delta G = \delta^2\Phi/\delta G\delta G$ . As shown by Baym<sup>17</sup>, any self-energy approximation is conserving whenever: (i) the self-energy can be written as the derivative of a functional  $\Phi[G]$ , i.e.  $\Sigma = \delta\Phi[G]/\delta G$ , and (ii) the SD equation for  $G$  needs to be solved fully self-consistently for this form of the self-energy. Second, the collective excitations of the Hubbard model can be calculated in two different ways: as poles of the fermion Green's function,  $K$ , and as poles of the boson Green's function,  $D$ ; or equivalently, as poles of the density and spin parts of the general response function,  $\Pi$ . Here, the boson Green's function,  $D$ , is defined by the equation  $D = D^{(0)} + D^{(0)}\Pi D^{(0)}$  where  $D^{(0)}$  is the free boson propagator. Third, the action which describes the interactions in the Hubbard model is similar to the action  $\psi^{\dagger}A\psi$  in quantum electrodynamics. This allows us to apply the powerful field-theoretical methods, such as the method of Legendre transforms, to derive the SD and BS equations, as well as the vertex equation for the vertex function,  $\Gamma$ , and the Dyson equation for the boson Green's function,  $D$ .

The basic assumption in our BS formalism is that the bound states of two fermions in the optical lattice at zero temperature are described by the BS wave functions (BS amplitudes). The BS amplitude determines the probability amplitude to find the first electron at the site  $i$  at the moment  $t_1$  and the second electron at the site  $j$  at the moment  $t_2$ . The BS amplitude depends on the relative internal time  $t_1 - t_2$  and on the center-of-mass time  $(t_1 + t_2)/2$ .<sup>18</sup>

$$\begin{aligned}\Phi_{n_2 n_1}^{\mathbf{Q}}(\mathbf{r}_i, \mathbf{r}_j; t_1, t_2) &= \\ \exp\{i[\mathbf{Q} \cdot (\mathbf{r}_i + \mathbf{r}_j)/2 - \omega(\mathbf{Q})(t_1 + t_2)/2]\} & \quad (3) \\ \Psi_{n_2 n_1}^{\mathbf{Q}}(\mathbf{r}_i - \mathbf{r}_j, t_1 - t_2),\end{aligned}$$

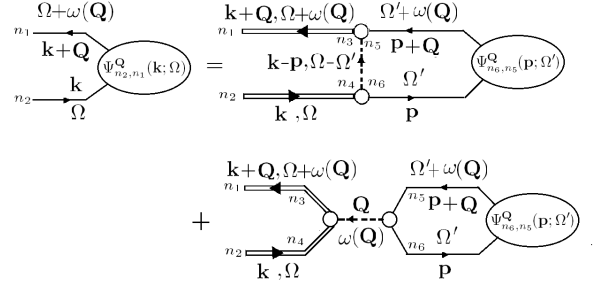


FIG. 1: Diagrammatic representation of the Bethe-Salpeter equation (4) for the BS amplitude  $\Psi_{n_2, n_1}^{\mathbf{Q}}(\mathbf{k}; \Omega)$ . The single-particle Green's function  $G_{n_2 n_1}(\mathbf{k}, \Omega)$  is denoted by two solid lines, oriented in the direction of electron propagation. The dashed lines represent the free boson propagator  $D_{\alpha\beta}^{(0)}(\omega(\mathbf{Q}), \Omega)$ . At each of the bare vertices  $\Gamma_{\alpha}^{(0)}(n_1, n_2)$  (represented by circles) the energy and momentum are conserved.

where  $\mathbf{Q}$  and  $\omega(\mathbf{Q})$  are the collective-mode momentum and the corresponding dispersion, respectively. Since  $\{n_1, n_2\} = \{1, 2, 3, 4\}$ , we have to take into account the existence of sixteen BS amplitudes. The Fourier transform of  $\Psi_{n_2 n_1}^{\mathbf{Q}}(\mathbf{r}, t) = \int \frac{d\Omega}{2\pi} \int \frac{d^d \mathbf{k}}{(2\pi)^d} e^{i(\mathbf{k} \cdot \mathbf{r} - \Omega t)} \Psi_{n_2, n_1}^{\mathbf{Q}}(\mathbf{k}; \Omega)$  satisfies the following BS equation, presented diagrammatically in FIG. 1:

$$\begin{aligned}\Psi_{n_2, n_1}^{\mathbf{Q}}(\mathbf{k}; \Omega) &= G_{n_1 n_3}(\mathbf{k} + \mathbf{Q}, \Omega + \omega(\mathbf{Q})) G_{n_4 n_2}(\mathbf{k}, \Omega) \int \frac{d\Omega'}{2\pi} \int \frac{d^d \mathbf{p}}{(2\pi)^d} \\ &\left[ I_d \begin{pmatrix} n_3 & n_5 \\ n_4 & n_6 \end{pmatrix} | \mathbf{k} - \mathbf{p}, \Omega - \Omega' \right] + I_{exc} \begin{pmatrix} n_3 & n_5 \\ n_4 & n_6 \end{pmatrix} | \mathbf{Q}, \omega(\mathbf{Q}) \right] \\ &\Psi_{n_6, n_5}^{\mathbf{Q}}(\mathbf{p}; \Omega').\end{aligned}\quad (4)$$

Here,  $G_{n_1 n_2}(\mathbf{k}, \Omega)$  is the single-particle Green's function, and  $I_d$  and  $I_{exc}$  are the direct and exchange parts of the BS kernel.

In a similar problem of interacting photons and electrons in electrodynamics, the direct part of the BS kernel does depend on frequency; therefore the solution of the BS equation is more complicated. As we shall see in the next Section, in the case of the  $t - U - J$  model the boson propagator  $D_{\alpha\beta}^{(0)}(\mathbf{k})$  ( $\alpha, \beta = 1, 2, 3, 4$ ) is frequency independent, and therefore, the following BS equation for the equal-time BS amplitude  $\Psi_{n_2, n_1}^{\mathbf{Q}}(\mathbf{k}) = \int \frac{d\Omega}{2\pi} \Psi_{n_2, n_1}^{\mathbf{Q}}(\mathbf{k}; \Omega)$  takes place:

$$\begin{aligned}\Psi_{n_2, n_1}^{\mathbf{Q}}(\mathbf{k}) &= \int \frac{d\Omega}{2\pi} G_{n_1 n_3}(\mathbf{k} + \mathbf{Q}, \Omega + \omega(\mathbf{Q})) G_{n_4 n_2}(\mathbf{k}, \Omega) \\ &\int \frac{d^d \mathbf{p}}{(2\pi)^d} [-\Gamma_{\alpha}^{(0)}(n_3, n_5) D_{\alpha\beta}^{(0)}(\mathbf{k} - \mathbf{p}) \Gamma_{\beta}^{(0)}(n_6, n_4) \\ &+ \frac{1}{2} \Gamma_{\alpha}^{(0)}(n_3, n_4) D_{\alpha\beta}^{(0)}(\mathbf{Q}) \Gamma_{\beta}^{(0)}(n_6, n_5)] \Psi_{n_6, n_5}^{\mathbf{Q}}(\mathbf{p}).\end{aligned}\quad (5)$$

In the case of  $t-U-J$  model the interaction in the direct kernel can be factorized, i.e.  $D_{\alpha\beta}^{(0)}(\mathbf{k}-\mathbf{p}) = f_\alpha(\mathbf{k})g_\beta(\mathbf{p})$ ; therefore it is possible to obtain the collective excitation spectrum using an  $80 \times 80$  secular determinant. Such an ambitious task will be left as a subject of future research. Instead, we shall discuss the limit  $J \rightarrow 0$  in which the dimensions of the secular determinant becomes  $16 \times 16$ .

The paper is organized as follows. In the next section we apply the functional-integral formalism to derive equations for the single-particle excitations and for the two-particle collective modes. In Section III, we numerically solve the BS equation in the case of 1D and 2D stationary and moving lattices. We use the mean-field approximation for the single-particle Green's functions, while the spectrum of the collective excitations is obtained in the GRPA. The technical details have been deferred to the Appendix.

## II. FIELD-THEORETICAL APPROACH TO THE COLLECTIVE MODES OF THE $t-U-J$ MODEL

### A. The functional-integral approach

The Green's functions in the field-theoretical approach are defined by means of the so-called generating functional with sources for the boson and fermion fields. In our problem the corresponding functional integrals cannot be evaluated exactly because the interaction part of the Hamiltonian (1) is quartic in the Grassmann fermion fields. However, we can transform the quartic terms to a quadratic form by introducing a model system which consists of a four-component boson field  $A_\alpha(z)$  ( $\alpha = 1, 2, 3, 4$ ) interacting with fermion fields  $\hat{\psi}(y)$

and  $\hat{\psi}(x)$ . The spin-dependent nature of the interactions requires four spin degrees of freedom of the Bose field  $A_\alpha(z)$ . The action of our model system is assumed to be of the following form  $S = S_0^{(e)} + S_0^{(A)} + S^{(e-A)}$ , where (throughout this paper we use the summation-integration convention: that repeated variables are summed up or integrated over):

$$S_0^{(e)} = \hat{\bar{\psi}}(y) \hat{G}^{(0)-1}(y; x) \hat{\psi}(x),$$

$$S_0^{(A)} = \frac{1}{2} A_\alpha(z) D_{\alpha\beta}^{(0)-1}(z, z') A_\beta(z'),$$

$$S^{(e-A)} = \hat{\bar{\psi}}(y) \hat{\Gamma}_\alpha^{(0)}(y, x | z) \hat{\psi}(x) A_\alpha(z).$$

The action  $S_0^{(e)}$  describes the fermion (electron) part of the system. The inverse Green's function of free electrons  $\hat{G}^{(0)-1}(y; x)$  is  $4 \times 4$  diagonal matrix:

$$\hat{G}^{(0)-1}(y; x) =$$

$$\sum_{\mathbf{k}, \omega_m} \exp[i\mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_{i'}) - \omega_m(u - u')] G_{n_1 n_2}^{(0)-1}(\mathbf{k}, \omega_m),$$

where  $G_{11}^{(0)-1}(\mathbf{k}, \omega_m) = -G_{33}^{(0)-1}(-\mathbf{k}, -\omega_m) = \omega_m - \xi(\mathbf{k})$ , and  $-G_{22}^{(0)-1}(-\mathbf{k}, -\omega_m) = G_{44}^{(0)-1}(\mathbf{k}, \omega_m) = \omega_m + \xi(\mathbf{k})$ . The symbol  $\sum_{\omega_m}$  is used to denote  $\beta^{-1} \sum_m$  (for fermion fields  $\omega_m = (2\pi/\beta)(m + 1/2)$ ;  $m = 0, \pm 1, \pm 2, \dots$ ), and  $\xi(\mathbf{k}) = 2t_x(1 - \cos k_x) + 2t_y(1 - \cos k_y) - \mu$  is the non-interacting dispersion on a square lattice.

The action  $S_0^{(A)}$  describes the boson field. The bare boson propagator in  $S_0^{(A)}$  provides the spin-dependent interactions  $U$  and  $J$ , and it is defined as:

$$\hat{D}^{(0)}(z, z') = \delta(v - v') \left[ U \delta_{j, j'} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \sum_a \delta_{j', j+a} \begin{pmatrix} \frac{1}{2}J - \frac{1}{2}J & 0 & 0 \\ \frac{1}{2}J & \frac{1}{2}J & 0 & 0 \\ 0 & 0 & 0 & J \\ 0 & 0 & J & 0 \end{pmatrix} \right].$$

Here the summation on  $a$  runs over the nearest-neighbor sites of site  $j$ . The Fourier transform of the boson propagator is given by

$$\begin{aligned} \hat{D}^{(0)}(z, z') &= \frac{1}{N} \sum_{\mathbf{k}} \sum_{\omega_p} e^{i[\mathbf{k} \cdot (\mathbf{r}_j - \mathbf{r}_{j'}) - \omega_p(v - v')]} \hat{D}^{(0)}(\mathbf{k}), \\ \hat{D}^{(0)}(\mathbf{k}) &= \begin{pmatrix} J(\mathbf{k}) & U - J(\mathbf{k}) & 0 & 0 \\ U - J(\mathbf{k}) & J(\mathbf{k}) & 0 & 0 \\ 0 & 0 & 0 & 2J(\mathbf{k}) \\ 0 & 0 & 2J(\mathbf{k}) & 0 \end{pmatrix} \end{aligned} \quad (6)$$

where the symbol  $\sum_{\omega_p}$  is used to denote  $\beta^{-1} \sum_p$  (for boson fields  $\omega_p = (2\pi/\beta)p$ ;  $p = 0, \pm 1, \pm 2, \dots$ ), and in the case of a two-dimensional square lattice  $J(\mathbf{k}) = J(\cos k_x + \cos k_y)$ .

The interaction between the fermion and the boson fields is described by the action  $S^{(e-A)}$ . The bare vertex

$\widehat{\Gamma}_\alpha^{(0)}(y_1; x_2 | z) = \widehat{\Gamma}_\alpha^{(0)}(i_1, u_1; i_2, u_2 | j, v) = \delta(u_1 - u_2)\delta(u_1 - v)\delta_{i_1 i_2}\delta_{i_1 j}\widehat{\Gamma}^{(0)}(\alpha)$  is a  $4 \times 4$  matrix, where

$$\widehat{\Gamma}^{(0)}(\alpha) = \frac{1}{2}(\gamma_0 + \alpha_z)\delta_{\alpha 1} + \frac{1}{2}(\gamma_0 - \alpha_z)\delta_{\alpha 2} + \frac{1}{2}(\alpha_x + i\alpha_y)\delta_{\alpha 3} + \frac{1}{2}(\alpha_x - i\alpha_y)\delta_{\alpha 4}. \quad (7)$$

The Dirac matrix  $\gamma_0$  and the  $4 \times 4$  matrices  $\widehat{\alpha}_i$  are defined as ( $\sigma_i$  are the Pauli matrices):

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \widehat{\alpha}_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_y \sigma_i \sigma_y \end{pmatrix}, i = x, y, z.$$

The relation between the  $t - U - J$  model and our model system can be demonstrated by applying the Hubbard-Stratonovich transformation for the electron operators:

$$\begin{aligned} & \int \mu[A] \exp \left[ \widehat{\psi}(y) \widehat{\Gamma}_\alpha^{(0)}(y; x|z) \widehat{\psi}(x) A_\alpha(z) \right] \\ &= \exp \left[ -\frac{1}{2} \widehat{\psi}(y) \widehat{\Gamma}_\alpha^{(0)}(y; x|z) \widehat{\psi}(x) D_{\alpha, \beta}^{(0)}(z, z') \widehat{\psi}(y') \widehat{\Gamma}_\beta^{(0)}(y'; x'|z') \widehat{\psi}(x') \right]. \end{aligned} \quad (8)$$

The functional measure  $D\mu[A]$  is chosen to be:

$$\mu[A] = D A e^{-\frac{1}{2} A_\alpha(z) D_{\alpha, \beta}^{(0)-1}(z, z') A_\beta(z')}, \int \mu[A] = 1.$$

The Hubbard-Stratonovich transformation allows us to map the  $t - U - J$  model onto the model system described by the action  $S$ . This transformation creates an extra term that can be included in the chemical potential  $\bar{\mu} = \mu - fU/2$ , and therefore, the mean-field expression for the chemical potential is recovered by the Hubbard-Stratonovich transformation. Since there exists a one-to-one correspondence between the  $t - U - J$  model and the model system defined by the action  $S$ , we can obtain the spectrum of the single-particle excitations, as well as the spectrum of the collective modes by analyzing the single-particle and two-particle excitations of the model system.

According to the field-theoretical approach, the expectation value of a general operator  $\widehat{O}(u)$  can be expressed as a functional integral over the boson field  $A$  and the Grassmann fermion fields  $\widehat{\psi}$  and  $\widehat{\bar{\psi}}$ :

$$\langle \widehat{T}_u(\widehat{O}(u)) \rangle = \frac{1}{Z[J, M]} \int D\mu[\widehat{\bar{\psi}}, \widehat{\psi}, A] \widehat{O}(u) \exp \left[ J_\alpha(z) A_\alpha(z) - M(\widehat{\bar{\psi}}, \widehat{\psi}) \right] |_{J=M=0}, \quad (9)$$

where the symbol  $\langle \dots \rangle$  means that the thermodynamic average is made, and  $\widehat{T}_u$  is an  $u$ -ordering operator. The functional  $Z[J, M]$  is defined by

$$Z[J, M] = \int D\mu[\widehat{\bar{\psi}}, \widehat{\psi}, A] \widehat{O}(u) \exp \left[ J_\alpha(z) A_\alpha(z) - M(\widehat{\bar{\psi}}, \widehat{\psi}) \right], \quad (10)$$

where the functional measure  $D\mu[\widehat{\bar{\psi}}, \widehat{\psi}, A] = D A D \widehat{\bar{\psi}} D \widehat{\psi} \exp(S)$  satisfies the condition  $\int D\mu[\widehat{\bar{\psi}}, \widehat{\psi}, A] = 1$ . The quantity  $J_\alpha(z)$  is the source of the boson field, while the sources  $M_{ij}(y; x)$  of the fermion fields are included in the  $M(\widehat{\bar{\psi}}, \widehat{\psi})$  term :

$$\begin{aligned} M(\widehat{\bar{\psi}}, \widehat{\psi}) &= \psi_\uparrow^\dagger(y) M_{11}(y; x) \psi_\uparrow(x) + \psi_\downarrow^\dagger(y) M_{21}(y; x) \psi_\uparrow(x) + \psi_\uparrow^\dagger(y) M_{12}(y; x) \psi_\downarrow(x) + \psi_\downarrow^\dagger(y) M_{22}(y; x) \psi_\downarrow(x) \\ &+ \psi_\uparrow(y) M_{31}(y; x) \psi_\uparrow^\dagger(x) + \psi_\downarrow(y) M_{41}(y; x) \psi_\uparrow^\dagger(x) + \psi_\uparrow(y) M_{32}(y; x) \psi_\downarrow^\dagger(x) + \psi_\downarrow(y) M_{42}(y; x) \psi_\downarrow^\dagger(x) \\ &+ \psi_\uparrow^\dagger(y) M_{13}(y; x) \psi_\uparrow^\dagger(x) + \psi_\downarrow^\dagger(y) M_{23}(y; x) \psi_\uparrow^\dagger(x) + \psi_\uparrow^\dagger(y) M_{14}(y; x) \psi_\downarrow^\dagger(x) + \psi_\downarrow^\dagger(y) M_{24}(y; x) \psi_\downarrow^\dagger(x) \\ &+ \psi_\uparrow(y) M_{33}(y; x) \psi_\uparrow^\dagger(x) + \psi_\downarrow(y) M_{43}(y; x) \psi_\uparrow^\dagger(x) + \psi_\uparrow(y) M_{34}(y; x) \psi_\downarrow^\dagger(x) + \psi_\downarrow(y) M_{44}(y; x) \psi_\downarrow^\dagger(x) \end{aligned} \quad (11)$$

Here, we have introduced complex indices  $1 = \{n_1, x_1\}$ , and  $2 = \{n_2, y_2\}$  where,  $x_1 = \{\mathbf{r}_{i_1}, u_1\}$ ,  $y_2 = \{\mathbf{r}_{i_2}, u_2\}$  and  $\{n_1, n_2\} = \{1, 2, 3, 4\}$ . We shall now use a functional derivative  $\delta/\delta M(2; 1)$ ; depending on the spin degrees of freedom, there are sixteen possible derivatives.

By means of the definition (9), one can express all Green's functions related to system under consideration in terms of the functional derivatives with respect to the corresponding sources of the generating functional of the connected Green's functions  $W[J, M] = \ln Z[J, M]$ . Thus, we define the following Green's and vertex functions which will be used to analyze the collective modes of our model:

Boson Green's function  $D_{\alpha\beta}(z, z')$ :

This function is a  $4 \times 4$  matrix defined as  $D_{\alpha\beta}(z, z') = -\frac{\delta^2 W}{\delta J_\alpha(z) \delta J_\beta(z')}$ .

Single-electron Green's function  $G_{n_1 n_2}(x_1; y_2)$ :

This function is a  $4 \times 4$  matrix whose elements are  $G_{n_1 n_2}(x_1; y_2) = -\delta W / \delta M_{n_2 n_1}(y_2; x_1)$ :

$$\hat{G}(x_1; y_2) = - \begin{pmatrix} \langle \hat{T}_u(\psi_\uparrow(x_1)\psi_\uparrow^\dagger(y_2)) \rangle & \langle \hat{T}_u(\psi_\uparrow(x_1)\psi_\downarrow^\dagger(y_2)) \rangle & \langle \hat{T}_u(\psi_\uparrow(x_1)\psi_\uparrow(y_2)) \rangle & \langle \hat{T}_u(\psi_\uparrow(x_1)\psi_\downarrow(y_2)) \rangle \\ \langle \hat{T}_u(\psi_\downarrow(x_1)\psi_\uparrow^\dagger(y_2)) \rangle & \langle \hat{T}_u(\psi_\downarrow(x_1)\psi_\downarrow^\dagger(y_2)) \rangle & \langle \hat{T}_u(\psi_\downarrow(x_1)\psi_\uparrow(y_2)) \rangle & \langle \hat{T}_u(\psi_\downarrow(x_1)\psi_\downarrow(y_2)) \rangle \\ \langle \hat{T}_u(\psi_\uparrow^\dagger(x_1)\psi_\uparrow^\dagger(y_2)) \rangle & \langle \hat{T}_u(\psi_\uparrow^\dagger(x_1)\psi_\downarrow^\dagger(y_2)) \rangle & \langle \hat{T}_u(\psi_\uparrow^\dagger(x_1)\psi_\uparrow(y_2)) \rangle & \langle \hat{T}_u(\psi_\uparrow^\dagger(x_1)\psi_\downarrow(y_2)) \rangle \\ \langle \hat{T}_u(\psi_\downarrow^\dagger(x_1)\psi_\uparrow^\dagger(y_2)) \rangle & \langle \hat{T}_u(\psi_\downarrow^\dagger(x_1)\psi_\downarrow^\dagger(y_2)) \rangle & \langle \hat{T}_u(\psi_\downarrow^\dagger(x_1)\psi_\uparrow(y_2)) \rangle & \langle \hat{T}_u(\psi_\downarrow^\dagger(x_1)\psi_\downarrow(y_2)) \rangle \end{pmatrix}. \quad (12)$$

Depending on the two spin degrees of freedom,  $\sigma_1$  and  $\sigma_2$ , there exist eight "normal" Green functions and eight "anomalous" Green's functions. We introduce Fourier transforms of the "normal"  $G_{\sigma_1, \sigma_2}(\mathbf{k}, u_1 - u_2) = -\langle \hat{T}_u(\psi_{\sigma_1, \mathbf{k}}(u_1)\psi_{\sigma_2, \mathbf{k}}^\dagger(u_2)) \rangle$ , and "anomalous"  $F_{\sigma_1, \sigma_2}(\mathbf{k}, u_1 - u_2) = -\langle \hat{T}_u(\psi_{\sigma_1, \mathbf{k}}(u_1)\psi_{\sigma_2, -\mathbf{k}}(u_2)) \rangle$  one-particle Green's functions, where  $\{\sigma_1, \sigma_2\} = \uparrow, \downarrow$ . Here  $\psi_{\uparrow, \mathbf{k}}^\dagger(u)$ ,  $\psi_{\uparrow, \mathbf{k}}(u)$  and  $\psi_{\downarrow, \mathbf{k}}^\dagger(u)$ ,  $\psi_{\downarrow, \mathbf{k}}(u)$  are the creation-annihilation Heisenberg operators. The Fourier transform of the single-particle Green's function is given by

$$\hat{G}(1; 2) = \frac{1}{N} \sum_{\mathbf{k}} \sum_{\omega_m} \exp\{i[\mathbf{k} \cdot (\mathbf{r}_{i_1} - \mathbf{r}_{i_2}) - \omega_m(u_1 - u_2)]\} \begin{pmatrix} \hat{G}(\mathbf{k}, \omega_m) & \hat{F}(\mathbf{k}, \omega_m) \\ \hat{F}^\dagger(\mathbf{k}, \omega_m) & -\hat{G}(-\mathbf{k}, -\omega_m) \end{pmatrix}. \quad (13)$$

Here,  $\hat{G}$  and  $\hat{F}$  are  $2 \times 2$  matrices whose elements are  $G_{\sigma_1, \sigma_2}$  and  $F_{\sigma_1, \sigma_2}$ , respectively.

The two-particle Green's function  $K \begin{pmatrix} n_1, x_1 & n_3, y_3 \\ n_2, y_2 & n_4, x_4 \end{pmatrix}$ :

For different  $n_1, n_2, n_3, n_4$ , there are 256 two-particle Green's functions:

$$K \begin{pmatrix} n_1, x_1 & n_3, y_3 \\ n_2, y_2 & n_4, x_4 \end{pmatrix} = K \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \frac{\delta^2 W}{\delta M_{n_2 n_1}(y_2; x_1) \delta M_{n_3 n_4}(y_3; x_4)} = -\frac{\delta G_{n_1 n_2}(x_1; y_2)}{\delta M_{n_3 n_4}(y_3; x_4)}; \quad (14)$$

This definition of  $K$  allows us to conclude that if the approximation used for  $G$  is chosen in accordance with the recipes proposed by Baym and Kadanoff,<sup>17</sup> then  $K$  is automatically conserving.

The vertex function  $\hat{\Gamma}_\alpha(2; 1 | z)$ :

For fixed  $\alpha$  the vertex function is a  $4 \times 4$  matrix whose elements are:

$$\hat{\Gamma}_\alpha(i_2, u_2; i_1, u_1 | v, j)_{n_2 n_1} = -\frac{\delta G_{n_1 n_2}^{-1}(i_2, u_2; i_1, u_1)}{\delta J_\beta(z')} D_{\beta\alpha}^{-1}(z', z). \quad (15)$$

## B. Equations of the boson and fermion Green's functions

It is well-known that the electron self-energy (electron mass operator)  $\hat{\Sigma}(1; 2)$  can be defined by means of the so-called SD equations. They can be derived using the fact that the measure  $D\mu[\bar{\psi}, \psi, A]$  is invariant under the translations  $\bar{\psi} \rightarrow \bar{\psi} + \delta\bar{\psi}$  and  $A \rightarrow A + \delta A$ :

$$D_{\alpha\beta}^{(0)-1}(z, z') R_\beta(z') + \frac{1}{2} Tr \left( \hat{G}(1; 2) \hat{\Gamma}_\alpha^{(0)}(2; 1 | z) \right) + J_\alpha(z) = 0, \quad (16)$$

$$\hat{G}^{-1}(1; 2) - \hat{G}^{(0)-1}(1; 2) + \hat{\Sigma}(1; 2) + \hat{M}(1; 2) = 0, \quad (17)$$

where  $R_\alpha(z) = \delta W / \delta J_\alpha(z)$  is the average boson field. The electron self-energy  $\hat{\Sigma}$  is a  $4 \times 4$  matrix which can be written as a sum of Hartree  $\hat{\Sigma}^H$  and Fock  $\hat{\Sigma}^F$  parts. The Hartree part is a diagonal matrix whose elements are:

$$\Sigma^H(i_1, u_1; i_2, u_2)_{n_1 n_2} = \frac{1}{2} \hat{\Gamma}_\alpha^{(0)}(i_1, u_1; i_2, u_2 | j, v)_{n_1 n_2} D_{\alpha\beta}^{(0)}(j, v; j', v') \hat{\Gamma}_\beta^{(0)}(i_3, u_3; i_4, u_4 | j', v')_{n_3 n_4} G_{n_4 n_3}(i_4, u_4; i_3, u_3). \quad (18)$$

The Fock part of the electron self-energy is given by:

$$\begin{aligned} \Sigma^F(i_1, u_1; i_2, u_2)_{n_1 n_2} &= -\hat{\Gamma}_\alpha^{(0)}(i_1, u_1; i_6, u_6 | j, v)_{n_1 n_6} D_{\alpha\beta}^{(0)}(j, v; j', v') \hat{\Gamma}_\beta^{(0)}(i_4, u_4; i_5, u_5 | j', v')_{n_4 n_5} \times \\ &K \begin{pmatrix} n_5, i_5, u_5 & n_3, i_3, u_3 \\ n_4, i_4, u_4 & n_6, i_6, u_6 \end{pmatrix} G_{n_3 n_2}^{-1}(i_3, u_3; i_2, u_2). \end{aligned} \quad (19)$$

The Fock part of the electron self-energy depends on the two-particle Green's function  $K$ , and therefore, the SD equations and the BS equation for  $K$  have to be solved self-consistently.

Our approach to the  $t - U - J$  model allows us to obtain exact equations of the Green's functions by using the field-theoretical technique. We now wish to return to our statement that the Green's functions are the thermodynamic average of the  $\hat{T}_u$ -ordered products of field operators. The standard procedure for calculating the Green's functions is to apply Wick's theorem. This enables us to evaluate the  $\hat{T}_u$ -ordered products of field operators as a perturbation expansion involving only wholly contracted field operators. These expansions can be summed formally to yield different equations of Green's functions. The main disadvantage of this procedure is that the validity of the equations must be verified diagram by diagram. For this reason we will use the method of Legendre transforms of the generating functional for connected Green functions.<sup>19</sup> By applying the same steps as in Ref. [8] we obtain the BS equation of the two-particle Green's function, the Dyson equation of the boson Green's function, and the vertex equation:

$$K^{-1} \begin{pmatrix} n_2, i_2, u_2 & n_3, i_3, u_3 \\ n_1, i_1, u_1 & n_4, i_4, u_4 \end{pmatrix} = K^{(0)-1} \begin{pmatrix} n_2, i_2, u_2 & n_3, i_3, u_3 \\ n_1, i_1, u_1 & n_4, i_4, u_4 \end{pmatrix} - I \begin{pmatrix} n_2, i_2, u_2 & n_3, i_3, u_3 \\ n_1, i_1, u_1 & n_4, i_4, u_4 \end{pmatrix}, \quad (20)$$

$$D_{\alpha\beta}(z, z') = D_{\alpha\beta}^{(0)}(z, z') + D_{\alpha\gamma}^{(0)}(z, z'') \Pi_{\gamma\delta}(z'', z''') D_{\delta\beta}^{(0)}(z, z'), \quad (21)$$

$$\begin{aligned} \hat{\Gamma}_\alpha(i_2, u_2; i_1, u_1 | z)_{n_2 n_1} &= \hat{\Gamma}_\alpha^{(0)}(i_2, u_2; i_1, u_1 | z)_{n_2 n_1} + I \begin{pmatrix} n_2, i_2, u_2 & n_3, i_3, u_3 \\ n_1, i_1, u_1 & n_4, i_4, u_4 \end{pmatrix} \times \\ &K^{(0)} \begin{pmatrix} n_3, i_3, u_3 & n_6, i_6, u_6 \\ n_4, i_4, u_4 & n_5, i_5, u_5 \end{pmatrix} \hat{\Gamma}_\alpha(i_6, u_6; i_5, u_5 | z)_{n_6 n_5}. \end{aligned} \quad (22)$$

Here,

$$K^{(0)} \begin{pmatrix} n_2, i_2, u_2 & n_3, i_3, u_3 \\ n_1, i_1, u_1 & n_4, i_4, u_4 \end{pmatrix} = G_{n_2 n_3}(i_2, u_2; i_3, u_3) G_{n_4 n_1}(i_4, u_4; i_1, u_1)$$

is the two-particle free propagator constructed from a pair of fully dressed single-particle Green's functions. The kernel  $I = \delta\Sigma/\delta G$  of the BS equation can be expressed as a functional derivative of the electron self-energy  $\hat{\Sigma}$ . Since  $\hat{\Sigma} = \hat{\Sigma}^H + \hat{\Sigma}^F$ , the BS kernel  $I = I_{exc} + I_d$  is a sum of functional derivatives of the Hartree  $\Sigma_H$  and Fock  $\Sigma_F$  contributions to the self-energy:

$$I_{exc} \begin{pmatrix} n_2, i_2, u_2 & n_3, i_3, u_3 \\ n_1, i_1, u_1 & n_4, i_4, u_4 \end{pmatrix} = \frac{\delta\Sigma^H(i_2, u_2; i_3, u_3)_{n_2 n_1}}{\delta G_{n_3 n_4}(i_3, u_3; i_4, u_4)}, \quad I_d \begin{pmatrix} n_2, i_2, u_2 & n_3, i_3, u_3 \\ n_1, i_1, u_1 & n_4, i_4, u_4 \end{pmatrix} = \frac{\delta\Sigma^F(i_2, u_2; i_3, u_3)_{n_2 n_1}}{\delta G_{n_3 n_4}(i_3, u_3; i_4, u_4)}. \quad (23)$$

The general response function  $\Pi$  in the Dyson equation (21) is defined as

$$\Pi_{\alpha\beta}(z, z') = \hat{\Gamma}_\alpha^{(0)}(i_1, u_1; i_2, u_2 | z)_{n_1 n_2} K \begin{pmatrix} n_2, i_2, u_2 & n_3, i_3, u_3 \\ n_1, i_1, u_1 & n_4, i_4, u_4 \end{pmatrix} \hat{\Gamma}_\beta^{(0)}(i_3, u_3; i_4, u_4 | z')_{n_3 n_4}. \quad (24)$$

The functions  $D$ ,  $K$  and  $\hat{\Gamma}$  are related by the identity:

$$\begin{aligned} &K^{(0)} \begin{pmatrix} n_2, i_2, u_2 & n_3, i_3, u_3 \\ n_1, i_1, u_1 & n_4, i_4, u_4 \end{pmatrix} \hat{\Gamma}_\beta(i_4, u_4; i_3, u_3 | z')_{n_4 n_3} D_{\beta\alpha}(z', z) \\ &= K \begin{pmatrix} n_2, i_2, u_2 & n_3, i_3, u_3 \\ n_1, i_1, u_1 & n_4, i_4, u_4 \end{pmatrix} \hat{\Gamma}_\beta^{(0)}(i_4, u_4; i_3, u_3 | z')_{n_4 n_3} D_{\beta\alpha}^{(0)}(z', z), \end{aligned} \quad (25)$$

By introducing the boson proper self-energy  $P_{\alpha\beta}^{-1}(z, z') = \Pi_{\alpha\beta}^{-1}(z, z') + D_{\alpha\beta}^{(0)}(z, z')$  one can rewrite the Dyson equation (21) for the boson Green's function as:

$$D_{\alpha\beta}^{-1}(z, z') = D_{\alpha\beta}^{(0)-1}(z, z') - P_{\alpha\beta}(z, z'). \quad (26)$$

The proper self-energy and the vertex function  $\hat{\Gamma}$  are related by the following equation:

$$\begin{aligned} P_{\alpha\beta}(z, z') &= \frac{1}{2} \text{Tr} \left[ \hat{\Gamma}_{\alpha}^{(0)}(y_1, x_2 | z) \hat{G}(x_2, y_3) \hat{\Gamma}_{\beta}(y_3, x_4 | z') \hat{G}(x_4, y_1) \right] \\ &= \frac{1}{2} \hat{\Gamma}_{\alpha}^{(0)}(i_1, u_1; i_2, u_2 | z)_{n_1 n_2} G_{n_2 n_3}(i_2, u_2; i_3, u_3) \hat{\Gamma}_{\beta}(i_3, u_3; i_4, u_4 | z')_{n_3 n_4} G_{n_4 n_1}(i_4, u_4; i_1, u_1). \end{aligned} \quad (27)$$

It is also possible to express the proper self-energy in terms of the two-particle Green's function  $\tilde{K}$  which satisfies the BS equation  $\tilde{K}^{-1} = K^{(0)-1} - I_d$ , but its kernel  $I_d = \delta \Sigma^F / \delta G$  includes only diagrams that represent the direct interactions:

$$\begin{aligned} P_{\alpha\beta}(z, z') &= \hat{\Gamma}_{\alpha}^{(0)}(i_1, u_1; i_2, u_2 | z)_{n_1 n_2} \tilde{K} \left( \begin{array}{cc} n_2, i_2, u_2 & n_3, i_3, u_3 \\ n_1, i_1, u_1 & n_4, i_4, u_4 \end{array} \right) \hat{\Gamma}_{\beta}^{(0)}(i_3, u_3; i_4, u_4 | z')_{n_3 n_4} \\ &= \hat{\Gamma}_{n_1 n_2}^{(0)}(\alpha) \tilde{K} \left( \begin{array}{cc} n_2, \mathbf{r}_j, v & n_3, \mathbf{r}_{j'}, v' \\ n_1, \mathbf{r}_j, v & n_4, \mathbf{r}_{j'}, v' \end{array} \right) \hat{\Gamma}_{n_3 n_4}^{(0)}(\beta). \end{aligned} \quad (28)$$

The proper boson self-energy  $\hat{P}$ , the vertex function  $\hat{\Gamma}$  and the two-particle Green's function  $\tilde{K}$  have common poles. Let  $\omega_{l\mathbf{Q}}$  and  $\mathbf{Q}$  denote the energy and momentum of one of these common poles. Close to  $\omega_{l\mathbf{Q}}$  one can write:

$$\tilde{K} \left( \begin{array}{cc} n_1, i_1, u_1 & n_3, i_3, u_3 \\ n_2, i_2, u_2 & n_4, i_4, u_4 \end{array} \right) \approx \sum_{\omega_p} e^{-i\omega_p(u_1 - u_3)} \frac{\psi_{n_2, n_1}^{l\mathbf{Q}}(\mathbf{r}_{i_2}, \mathbf{r}_{i_1}; u_2 - u_1) \psi_{n_3, n_4}^{l\mathbf{Q}*}(\mathbf{r}_{i_3}, \mathbf{r}_{i_4}; u_4 - u_3)}{i\omega_p - \omega_{l\mathbf{Q}}}, \quad (29)$$

where the amplitudes  $\psi_{n_2, n_1}^{l\mathbf{Q}}(\mathbf{r}_{i_2}, \mathbf{r}_{i_1}; u_2 - u_1)$  have the following form:

$$\psi_{n_2, n_1}^{l\mathbf{Q}}(\mathbf{r}_{i_2}, \mathbf{r}_{i_1}; u_2 - u_1) = \exp[i\mathbf{Q} \cdot (\mathbf{r}_{i_1} + \mathbf{r}_{i_2})/2] \psi_{n_2, n_1}^{l\mathbf{Q}}(\mathbf{r}_{i_2} - \mathbf{r}_{i_1}; u_2 - u_1).$$

Due to the definition of the bare vertex  $\hat{\Gamma}^{(0)}$ , we have to take into account only the equal "time"  $u_1 = u_2$  amplitudes:

$$\psi_{n_2, n_1}^{l\mathbf{Q}}(\mathbf{r}_{i_2} - \mathbf{r}_{i_1}; 0) = \frac{1}{N} \sum_{\mathbf{k}} \exp\{i\mathbf{k} \cdot (\mathbf{r}_{i_2} - \mathbf{r}_{i_1})\} \psi_{n_2, n_1}^{l\mathbf{Q}}(\mathbf{k}),$$

where  $\psi_{n_2, n_1}^{l\mathbf{Q}}(\mathbf{k})$  is the equal "time" two-particle wave functions in  $\mathbf{k}$ -representation. By means of (29) and (28) we obtain:

$$P_{\alpha\beta}(\mathbf{Q}, \omega) = \sum_l \left[ \frac{\varphi_{\alpha}^{l\mathbf{Q}} \varphi_{\beta}^{*l\mathbf{Q}}}{\omega - \omega_{l\mathbf{Q}} + i0^+} - \frac{\varphi_{\alpha}^{*l\mathbf{Q}} \varphi_{\beta}^{l\mathbf{Q}}}{\omega + \omega_{l\mathbf{Q}} + i0^+} \right], \quad (30)$$

where  $\omega_{l\mathbf{Q}} = E_l(\mathbf{Q}) - \bar{\mu}$ , and  $\varphi_{\alpha}^{l\mathbf{Q}} = \delta_{\alpha 1} (\psi_{1,1}^{l\mathbf{Q}}(0; 0) - \psi_{3,3}^{l\mathbf{Q}}(0; 0)) + \delta_{\alpha 2} (\psi_{2,2}^{l\mathbf{Q}}(0; 0) - \psi_{4,4}^{l\mathbf{Q}}(0; 0)) + \delta_{\alpha 3} (\psi_{1,2}^{l\mathbf{Q}}(0; 0) - \psi_{4,3}^{l\mathbf{Q}}(0; 0)) + \delta_{\alpha 4} (\psi_{2,1}^{l\mathbf{Q}}(0; 0) - \psi_{3,4}^{l\mathbf{Q}}(0; 0))$ . The Green's function (29) is invariant under the exchange  $4 \leftrightarrow 1$  and  $2 \leftrightarrow 3$ , therefore,  $\varphi_1^{l\mathbf{Q}} = -\varphi_2^{l\mathbf{Q}}$ , and  $\varphi_3^{l\mathbf{Q}} = -\varphi_4^{l\mathbf{Q}}$ . Thus, we obtain that  $P_{11} = P_{22}$ , and  $P_{33} = P_{44}$ . This means that the proper self-energy has a maximum of six different elements:

$$P_{\alpha\beta}(\mathbf{Q}; \omega) = \begin{pmatrix} P_{11}(\mathbf{Q}; \omega) & P_{12}(\mathbf{Q}; \omega) & P_{13}(\mathbf{Q}; \omega) & P_{14}(\mathbf{Q}; \omega) \\ P_{12}(\mathbf{Q}; \omega) & P_{11}(\mathbf{Q}; \omega) & P_{14}(\mathbf{Q}; \omega) & P_{13}(\mathbf{Q}; \omega) \\ P_{13}(\mathbf{Q}; \omega) & P_{14}(\mathbf{Q}; \omega) & P_{33}(\mathbf{Q}; \omega) & P_{34}(\mathbf{Q}; \omega) \\ P_{14}(\mathbf{Q}; \omega) & P_{13}(\mathbf{Q}; \omega) & P_{34}(\mathbf{Q}; \omega) & P_{33}(\mathbf{Q}; \omega) \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ b & a & d & c \\ c & d & f & h \\ d & c & h & f \end{pmatrix}. \quad (31)$$

### C. Spin-singlet order parameter

As we have already mentioned, the BS equation and the SD equations have to be solved self-consistently. The so-called  $D^{(0)}\Gamma^{(0)}$  approximation allows us to decouple the above-mentioned equations and to obtain a linearized integral equation for the Fock term. To apply this approximation we first use Eq. (25) to rewrite the Fock term as

$$\Sigma^F(i_1, u_1; i_2, u_2)_{n_1 n_2} = -\hat{\Gamma}_{\alpha}^{(0)}(i_1, u_1; i_3, u_3 | j, v)_{n_1 n_3} D_{\alpha\beta}(j, v; j', v')_{n_3 n_4} G_{n_3 n_4}(i_3, u_3; i_4, u_4) \hat{\Gamma}_{\beta}(i_4, u_4; i_2, u_2 | j', v')_{n_4 n_2}, \quad (32)$$



and after that we replace  $D$  and  $\hat{\Gamma}$  in (32) by the free boson propagator  $D^{(0)}$  and by the bare vertex  $\hat{\Gamma}^{(0)}$ , respectively. In this approximation the Fock term assumes the form:

$$\Sigma_0^F(i_1, u_1; i_2, u_2)_{n_1 n_2} = -\hat{\Gamma}_\alpha^{(0)}(i_1, u_1; i_3, u_3 | j, v)_{n_1 n_3} D_{\alpha\beta}^{(0)}(j, v; j', v') \hat{\Gamma}_\beta^{(0)}(i_4, u_4; i_2, u_2 | j', v')_{n_4 n_2} G_{n_3 n_4}(i_3, u_3; i_4, u_4) \quad (33)$$

As in the case of the Hartree self-energy, the contributions to  $\Sigma^F(i_1, u_1; i_2, u_2)$  due to the elements on the major diagonal of the above matrices will be included into the chemical potential  $\bar{\mu}$ . To obtain analytical results, we will assume two more approximations. First, we neglect  $G_{12} = G_{21} = G_{34} = G_{43} = 0$ , and second, we neglect the frequency dependence of the Fourier transform of the Fock part of the electron self-energy, i.e.  $\hat{\Sigma}^F(\mathbf{k}, i\omega_m) \approx \hat{\Delta}(\mathbf{k})$ , where the order parameter  $\hat{\Delta}$  is a  $4 \times 4$  matrix. It is known that in order to preserve the antisymmetry in the case of a spin-singlet Cooper-pairing, the order parameter  $\Delta(\mathbf{k})$  must be an even function of  $\mathbf{k}$ , so the momentum dependence of  $\hat{\Sigma}^F(\mathbf{k})$  will be

$$\hat{\Delta}(\mathbf{k}) = \begin{pmatrix} 0 & 0 & 0 & \Delta(\mathbf{k}) \\ 0 & 0 & -\Delta(\mathbf{k}) & 0 \\ 0 & -\Delta(\mathbf{k}) & 0 & 0 \\ \Delta(\mathbf{k}) & 0 & 0 & 0 \end{pmatrix}. \quad (34)$$

In this approximation the Fourier transform of the single-particle Green's function assumes the following form:

$$\hat{G}(\mathbf{k}, i\omega_m) = \frac{1}{(i\omega_m)^2 - E^2(\mathbf{k})} \begin{pmatrix} i\omega_m + \xi(\mathbf{k}) & 0 & 0 & \Delta(\mathbf{k}) \\ 0 & i\omega_m + \xi(\mathbf{k}) & -\Delta(\mathbf{k}) & 0 \\ 0 & -\Delta(\mathbf{k}) & i\omega_m - \xi(\mathbf{k}) & 0 \\ \Delta(\mathbf{k}) & 0 & 0 & i\omega_m - \xi(\mathbf{k}) \end{pmatrix}, \quad (35)$$

where  $E(\mathbf{k}) = \sqrt{\xi^2(\mathbf{k}) + \Delta(\mathbf{k})^2}$  and  $\xi(\mathbf{k}) = 2t_x(1 - \cos k_x) + 2t_y(1 - \cos k_y) - \bar{\mu}$ . Substituting this single-particle Green's function in Eq. (33) we reobtain the gap equation:

$$\Delta(\mathbf{k}) = \frac{1}{N} \sum_{\mathbf{q}} [-U + 3J(\mathbf{k} - \mathbf{q})] \frac{\Delta(\mathbf{q})}{2E(\mathbf{q})} \tanh\left(\frac{\beta E(\mathbf{q})}{2}\right). \quad (36)$$

In the case of the s-wave superfluidity we have  $J \rightarrow 0$ ,  $\Delta(\mathbf{k}) = \Delta$  and  $U < 0$ , and by using the above single-particle Green's function we obtain the gap equation for  $\Delta$  and the particle number equation for the filling factor:

$$f = \frac{2}{N} \sum_{\mathbf{k}} [u^2(\mathbf{k})f(E(\mathbf{k})) + v^2(\mathbf{k})f(-E(\mathbf{k}))], \quad (37)$$

$$1 = \frac{|U|}{N} \sum_{\mathbf{k}} \frac{f(-E(\mathbf{k})) - f(E(\mathbf{k}))}{2E_{\mathbf{p}}(\mathbf{k})},$$

$f(x) = [\exp(\beta x) + 1]^{-1}$  is the Fermi distribution function, and the coherence factors are  $u^2(\mathbf{k}) = [1 + \xi(\mathbf{k})/E(\mathbf{k})]/2$ ,  $v^2(\mathbf{k}) = 1 - u^2(\mathbf{k})$ .

The gap equation and the particle number equation in the case of a moving lattice are given in the Appendix.

#### D. Collective-mode equations

It is easy to see that in the case of a spin-singlet pairing  $a - b = h$  and  $c = d = f = 0$ . From the Dyson equation (26), it follows that the spectrum of the collective excitations  $\omega(\mathbf{Q})$  could be obtained by setting the  $4 \times 4$  determinant  $\det[D_{\alpha\beta}^{(0)-1}(\mathbf{Q}; \omega) - P_{\alpha\beta}(\mathbf{Q}; \omega)] = 0$ , or

$\det[\delta_{\alpha\beta} - D_{\alpha\gamma}^{(0)}(\mathbf{Q}; \omega) P_{\gamma\beta}(\mathbf{Q}; \omega)] = 0$ . Thus, we obtain the following exact equations for the collective modes:

$$0 = 1 + [U - 2J(\mathbf{Q})](a - b), \quad (38)$$

$$0 = 1 - U(a + b), \quad (39)$$

$$0 = 1 - 2J(\mathbf{Q})(a + b), \quad (40)$$

Since the equations (21) and (24) hold, the Fourier transforms of the general response function (24), the two-particle Green's function  $K$ , as well as the Fourier transform of the boson  $D_{\alpha\beta}$  Green's function should share the poles defined by (38) - (40).

To separate the spin and charge contributions to the poles of the general response function we express  $\Pi_{\alpha\beta}(\mathbf{Q}; \omega)$  in terms of the Fourier transform of the proper self-energy  $P_{\alpha\beta}(\mathbf{Q}; \omega)$ :

$$\Pi_{\alpha\beta}^{-1}(\mathbf{Q}; \omega) = P_{\alpha\beta}^{-1}(\mathbf{Q}; \omega) - D_{\alpha\beta}^{(0)}(\mathbf{Q}; \omega). \quad (41)$$

By means of (41) we obtain

$$\hat{\Pi}(\mathbf{Q}; \omega) = \hat{\Pi}^c(\mathbf{Q}; \omega) + \hat{\Pi}^s(\mathbf{Q}; \omega),$$

where  $\Pi_{\alpha\beta}^s(\mathbf{Q}; \omega)$  and  $\Pi_{\alpha\beta}^c(\mathbf{Q}; \omega)$  represent the contributions to the general response function due to the spin and

charge fluctuations:

$$\hat{\Pi}^s = \begin{pmatrix} \Pi_s^{(1)}(\mathbf{Q}; \omega) & -\Pi_s^{(1)}(\mathbf{Q}; \omega) & 0 & 0 \\ -\Pi_s^{(1)}(\mathbf{Q}; \omega) & \Pi_s^{(1)}(\mathbf{Q}; \omega) & 0 & 0 \\ 0 & 0 & 0 & \Pi_s^{(2)}(\mathbf{Q}; \omega) \\ 0 & 0 & \Pi_s^{(2)}(\mathbf{Q}; \omega) & 0 \end{pmatrix}, \quad (42)$$

$$\hat{\Pi}^c = \begin{pmatrix} \Pi_c(\mathbf{Q}; \omega) & \Pi_c(\mathbf{Q}; \omega) & 0 & 0 \\ \Pi_c(\mathbf{Q}; \omega) & \Pi_c(\mathbf{Q}; \omega) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (43)$$

where

$$\begin{aligned} \Pi_s^{(1)}(\mathbf{Q}; \omega) &= \frac{(a-b)/2}{1 + [U - 2J(\mathbf{Q})](a-b)}, \\ \Pi_s^{(2)}(\mathbf{Q}; \omega) &= \frac{a-b}{1 - 2J(\mathbf{Q})(a-b)}, \\ \Pi_c(\mathbf{Q}; \omega) &= \frac{(a+b)/2}{1 + U(a+b)}. \end{aligned} \quad (44)$$

The collective modes originating from spin fluctuations manifest themselves as poles of the following spin response functions (spin susceptibilities):

$$\begin{aligned} \chi_1^s(\mathbf{Q}; \omega) &= 2\Pi_s^{(1)}(\mathbf{Q}; \omega) = \frac{\kappa^s(\mathbf{Q}; \omega)}{1 + [U - 2J(\mathbf{Q})]\kappa^s(\mathbf{Q}; \omega)}, \\ \chi_2^s(\mathbf{Q}; \omega) &= \Pi_s^{(2)}(\mathbf{Q}; \omega) = \frac{\kappa^s(\mathbf{Q}; \omega)}{1 - 2J(\mathbf{Q})\kappa^s(\mathbf{Q}; \omega)} \end{aligned} \quad (45)$$

where  $\kappa^s(\mathbf{Q}; \omega) = a - b$ . The second collective mode, defined by Eq. (39), manifests itself as a pole of the charge density (charge susceptibility):

$$\chi^c(\mathbf{Q}; \omega) = 2\Pi_c(\mathbf{Q}; \omega) = \frac{\kappa^c(\mathbf{Q}; \omega)}{1 + 2V(\mathbf{Q})\kappa^c(\mathbf{Q}; \omega)}, \quad (46)$$

where  $\kappa^c(\mathbf{Q}; \omega) = a + b$ .

### E. Random phase approximation

The collective modes can be obtained from Eqs. (38) - (40) by solving the BS equation for the two-particle Green's function  $\tilde{K}$ . The solutions of the BS equation are not needed in the RPA, in which  $\tilde{K} \simeq K^{(0)}$ ; or, equivalently, in Eq. (27) for the proper self-energy we replace  $\hat{\Gamma}$  by  $\hat{\Gamma}^{(0)}$ :

$$\begin{aligned} P_{\alpha\beta}^{RPA}(z; z') &= \\ \frac{1}{2} Tr \left[ \hat{\Gamma}_{\alpha}^{(0)}(y, x|z) \hat{G}(x, y') \hat{\Gamma}_{\beta}^{(0)}(y', x'|z') \hat{G}(x', y) \right]. \end{aligned} \quad (47)$$

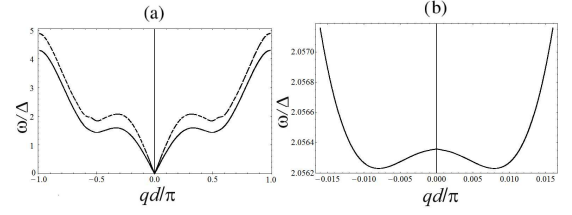


FIG. 2: Excitation spectra in 1D stationary optical lattice. We set  $t_x = t$ ,  $f = 0.5$ ,  $U = 2t$ . The superfluid gap and the chemical potential are  $\Delta = 0.409t$  and  $\mu = 0.624t$ . The solid line in (a) represents the spectrum obtained by solving the equation  $F_1(\omega, q) = 0$ , while the dashed line is plotted using  $F_2(\omega, q) = 0$ . The spectrum (b) represents the solution of the equation  $F_3(\omega, q) = 0$ .

The RPA expression for the proper self-energy provides  $a - b = h = \kappa_{zz}^{(0)}(\mathbf{Q}; \omega_p) = \kappa_{xx}^{(0)}(\mathbf{Q}; \omega_p) = \kappa_{yy}^{(0)}(\mathbf{Q}; \omega_p) = \kappa_0^s(\mathbf{Q}; \omega_p)$ ,  $a + b = \kappa_0^c(\mathbf{Q}; \omega_p)$  and  $c = d = f = 0$ , where the general susceptibility  $\kappa_{ij}^{(0)}(\mathbf{Q}; \omega_p)$ ,  $i, j = x, y, z$  was introduced decades ago:<sup>20</sup>

$$\begin{aligned} \kappa_{ij}^{(0)}(\mathbf{Q}; \omega_p) &= \\ \frac{1}{2} \sum_{\omega_m} \sum_{\mathbf{k}} Tr \left\{ \hat{\alpha}_i \hat{G}(\mathbf{k}; \omega_m) \hat{\alpha}_j \hat{G}(\mathbf{k} + \mathbf{Q}; \omega_p + \omega_m) \right\}. \end{aligned} \quad (48)$$

The noninteracting charge and spin susceptibilities  $\kappa_0^s(\mathbf{Q}; \omega)$  and  $\kappa_0^c(\mathbf{Q}; \omega)$  are:

$$\begin{aligned} \left\{ \begin{array}{l} \kappa_0^s(\mathbf{q}, \omega_p) \\ \kappa_0^c(\mathbf{q}, \omega_p) \end{array} \right\} &= \\ \frac{1}{N} \sum_{\mathbf{k}} \left\{ \begin{array}{l} \gamma_{\mathbf{k}, \mathbf{q}}^2 \\ l_{\mathbf{k}, \mathbf{q}}^2 \end{array} \right\} \frac{\zeta(\mathbf{k}, \mathbf{q}) [f(E(\mathbf{k})) - f(E(\mathbf{k} + \mathbf{q}))]}{(i\omega_p)^2 - \zeta^2(\mathbf{k}, \mathbf{q})} \\ + \frac{1}{N} \sum_{\mathbf{k}} \left\{ \begin{array}{l} \tilde{\gamma}_{\mathbf{k}, \mathbf{q}}^2 \\ m_{\mathbf{k}, \mathbf{q}}^2 \end{array} \right\} \frac{\varepsilon(\mathbf{k}, \mathbf{q}) [1 - f(E(\mathbf{k})) - f(E(\mathbf{k} + \mathbf{q}))]}{(i\omega_p)^2 - \varepsilon^2(\mathbf{k}, \mathbf{q})}. \end{aligned} \quad (49)$$

Here,  $\zeta(\mathbf{k}, \mathbf{q}) = E(\mathbf{k} + \mathbf{q}) - E(\mathbf{k})$ ,  $\varepsilon(\mathbf{k}, \mathbf{q}) = E(\mathbf{k} + \mathbf{q}) + E(\mathbf{k})$ , and the following form factors have been used:

$$\begin{aligned} \gamma_{\mathbf{k}, \mathbf{q}} &= u_{\mathbf{k}} u_{\mathbf{k}+\mathbf{q}} + v_{\mathbf{k}} v_{\mathbf{k}+\mathbf{q}}, & l_{\mathbf{k}, \mathbf{q}} &= u_{\mathbf{k}} u_{\mathbf{k}+\mathbf{q}} - v_{\mathbf{k}} v_{\mathbf{k}+\mathbf{q}}, \\ \tilde{\gamma}_{\mathbf{k}, \mathbf{q}} &= u_{\mathbf{k}} v_{\mathbf{k}+\mathbf{q}} - u_{\mathbf{k}+\mathbf{q}} v_{\mathbf{k}}, & m_{\mathbf{k}, \mathbf{q}} &= u_{\mathbf{k}} v_{\mathbf{k}+\mathbf{q}} + u_{\mathbf{k}+\mathbf{q}} v_{\mathbf{k}}. \end{aligned} \quad (50)$$

### III. COLLECTIVE MODES OF STATIONARY AND MOVING OPTICAL LATTICES AT ZERO TEMPERATURE

In the case of deep optical lattices we neglect the AF interaction, and we assume an attractive Hubbard interaction. The collective spectrum is obtained by solving

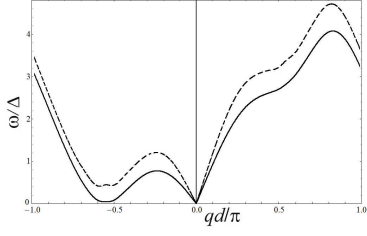


FIG. 3: Excitation spectra in 1D moving optical lattice ( $p=0.21$ ). We set  $t_x = t$ ,  $f = 0.5$ ,  $U = 2t$ . The superfluid gap and the chemical potential are  $\Delta = 0.420t$  and  $\mu = 0.655t$ . The solid line represents the spectrum obtained by solving the equation  $F_1(\omega, q) = 0$ , while the dashed line is plotted using  $F_2(\omega, q) = 0$ .

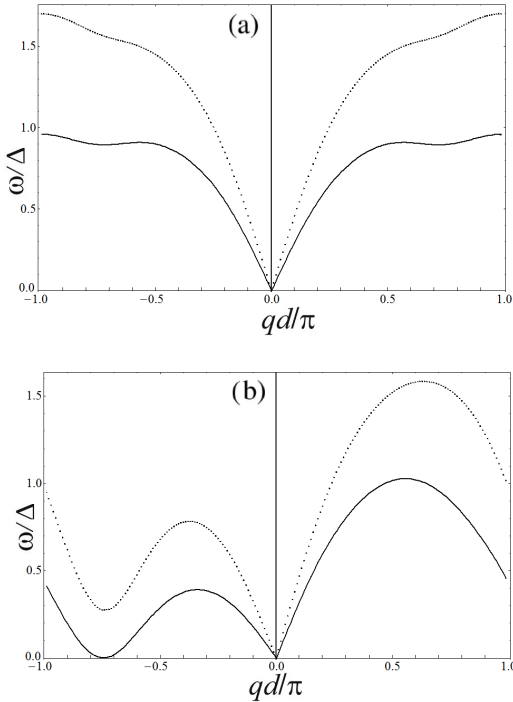


FIG. 4: Excitation spectra in 2D optical lattices for stationary (a) and moving (b) ( $p_x = p_y = 0.467/\sqrt{2}$ ) lattices. We set  $t_x = t_y = t$ ,  $f = 0.5$ ,  $U = 4.5t$ . The superfluid gap and chemical potential in the case of a stationary lattice are  $\Delta = 1.334t$  and  $\mu = 2.234t$ . For a moving lattice we have  $\Delta = 1.386t$  and  $\mu = 2.287t$ . The solid lines in (a) and (b) represent the spectrum obtained by solving the equation  $F_1(\omega, q) = 0$ , while the dotted lines are plotted using  $F_2(\omega, q) = 0$ .

the BS equation at zero temperature in the GRPA. In this approximation the corresponding equation for the BS amplitude  $\Psi_{n_2, n_1}^{\mathbf{Q}} = \int \frac{d\Omega}{2\pi} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \Psi_{n_2, n_1}^{\mathbf{Q}}(\mathbf{k}; \Omega)$  can be obtained from Eq. (5) by performing integration over the momentum vectors:

$$\Psi_{n_2 n_1}^{\mathbf{Q}} = K^{(0)} \begin{pmatrix} n_1 & n_3 \\ n_2 & n_4 \end{pmatrix} |\omega(\mathbf{Q}) \rangle \times \left[ I_d \begin{pmatrix} n_3 & n_5 \\ n_4 & n_6 \end{pmatrix} + I_{exc} \begin{pmatrix} n_3 & n_5 \\ n_4 & n_6 \end{pmatrix} \right] \Psi_{n_6, n_5}^{\mathbf{Q}}, \quad (51)$$

where the two-particle propagator  $K^{(0)}$  and the direct and exchange interactions are defined as follows:

$$K^{(0)} \begin{pmatrix} n_1 & n_3 \\ n_2 & n_4 \end{pmatrix} |\omega(\mathbf{Q}) \rangle \equiv K_{n_1 n_3 n_4 n_2} = \int \frac{d\Omega}{2\pi} \int \frac{d^d \mathbf{k}}{(2\pi)^d} G_{n_1 n_3}(\mathbf{k} + \mathbf{Q}, \Omega + \omega(\mathbf{Q})) G_{n_4 n_2}(\mathbf{k}, \Omega),$$

$$I_d \begin{pmatrix} n_1 & n_3 \\ n_2 & n_4 \end{pmatrix} = -\Gamma_{\alpha}^{(0)}(n_1, n_3) D_{\alpha\beta}^{(0)} \Gamma_{\beta}^{(0)}(n_4, n_2),$$

$$I_d \begin{pmatrix} n_1 & n_3 \\ n_2 & n_4 \end{pmatrix} = \frac{1}{2} \Gamma_{\alpha}^{(0)}(n_1, n_2) D_{\alpha\beta}^{(0)} \Gamma_{\beta}^{(0)}(n_4, n_3). \quad (52)$$

The boson propagator  $D_{\alpha\beta}^{(0)}$  in (52) is given by (6) in the limit  $J(\mathbf{k}) = 0$ .

The non-trivial solution of the BS equations (51) exists if the secular determinate  $\det[\hat{I} + U\hat{Z}]$  is zero, where  $\hat{I}$  is the unit matrix, and the  $16 \times 16$  matrix  $\hat{Z}$  is given in the Appendix. After computing the secular determinant, we find that at a given point  $\mathbf{Q}$  there exist three different dispersions  $\omega_j(\mathbf{Q})$ ,  $j = 1, 2, 3$ , which are the solutions of the three equations  $F_j(\omega_j, \mathbf{Q}) = 0$ , where:

$$F_1(\omega, \mathbf{Q}) = 1 + (I_{l,l} + I_{m,m} + I_{\gamma,\gamma})U + (-I_{l,m}^2 + I_{l,l}I_{m,m} - J_{\gamma,l}^2 - I_{\gamma,m}^2 + I_{l,l}I_{\gamma,\gamma} + I_{m,m}I_{\gamma,\gamma})U^2 + (-I_{m,m}J_{\gamma,l}^2 + 2I_{l,m}J_{\gamma,l}I_{\gamma,m} - I_{l,l}I_{\gamma,m}^2 - I_{\gamma,\gamma}I_{l,m}^2 + I_{l,l}I_{m,m}I_{\gamma,\gamma})U^3 \quad (53)$$

$$F_2(\omega, \mathbf{Q}) = 1 + (-I_{m,m} + 2I_{\gamma,\gamma})U + (-I_{l,l}^2 + J_{l,\tilde{\gamma}}^2 - I_{l,l}I_{m,m} - J_{\gamma,l}^2 + I_{l,l}I_{\gamma,\gamma} - I_{m,m}I_{\gamma,\gamma} + I_{\gamma,\gamma}^2 + I_{\gamma,\tilde{\gamma}}^2)U^2 + (I_{l,l}J_{\gamma,l}^2 + I_{m,m}J_{\gamma,l}^2 - I_{\gamma,\gamma}I_{l,l}^2 + I_{\gamma,\gamma}J_{l,\tilde{\gamma}}^2 - I_{\gamma,\gamma}I_{l,l}I_{m,m} - I_{\gamma,\gamma}J_{\gamma,l}^2 + I_{l,l}I_{\gamma,\gamma}^2 - 2J_{l,\tilde{\gamma}}J_{\gamma,l}I_{\gamma,\tilde{\gamma}} + I_{l,l}I_{\gamma,\tilde{\gamma}}^2)U^3 \quad (54)$$

$$F_3(\omega, \mathbf{Q}) = 1 + (-I_{l,l} - 2I_{m,m} + I_{\gamma,\gamma})U + (I_{l,l}I_{m,m} + I_{m,m}^2 - J_{\gamma,m}^2 - I_{\gamma,\gamma}I_{m,m})U^2 \quad (55)$$

The definitions of  $I_{a,b}$  and  $J_{a,b}$  are given in the Appendix, and we have used the relation  $I_{\tilde{\gamma},\tilde{\gamma}} = I_{l,l} + I_{m,m} - I_{\gamma,\gamma}$ .

The dispersion of the first rotonlike mode  $\omega_1(\mathbf{Q})$ , obtained by solving numerically the equation  $F_1(\omega_1, \mathbf{Q}) = 0$  has been previously obtained by the equation-of-motion method,<sup>4</sup> and by locating the poles of the density response function.<sup>5</sup> The numerical solution of the equation  $F_2(\omega_2, \mathbf{Q}) = 0$  provides the existence of a second rotonlike collective mode  $\omega_2(\mathbf{Q})$ . The two rotonlike modes have different low-energy Goldstone dispersions and different rotonlike minima.

In FIG. 2 - FIG. 4 we have plotted the excitation spectrum of superfluid Fermi gases in 1D and 2D optical lattices assuming the same strength of the interactions and filling factors as in FIG. 4 ( $U = 2t, f = 0.5$ ) and as in FIG. 9 ( $U = 4.5t, f = 0.5$ ) in Ref. [5]. In the presence of superfluid flow, as the flow momentum increases, the two rotonlike spectra lean toward the left side and the energies of the two rotonlike minima decrease. It is worth mentioning that the two rotonlike modes lie outside of the region determined by the lower boundary of the particle-hole continuum; therefore the two modes are not damped and constitute propagating modes and they should be experimentally observable. At a certain flow momentum, the minimum of the first rotonlike mode reaches zero energy, but this occurs before both the minimum of the second mode, and the lower boundary of the particle-hole continuum do. At this critical flow momentum the first mode is destabilized due to the spontaneous emission of rotonlike excitations; but to destabilize the second mode

one has to increase the flow momentum. Thus, there exist two critical lattice velocities which are determined by the existence of two rotonlike modes.

The third mode,  $\omega_3(\mathbf{Q})$ , is obtained by solving the equation  $F_3(\omega_3, \mathbf{Q}) = 0$ . This mode lies entirely inside the particle-hole continuum. Therefore, it is a non-propagating mode and cannot be an experimentally observable mode.

#### IV. CONCLUSION

In conclusion, we have studied the superfluid state of Fermi gases in deep optical lattices. Using the Bethe-Salpeter equation in the GRPA we have obtained the collective mode spectrum of the attractive Hubbard model in the presence of superfluid flow. We found that the spectrum of the collective excitations has two rotonlike modes with different low-energy Goldstone dispersions and different rotonlike minima. Both modes are experimentally observable because they are separated from the region determined by the lower boundary of the particle-hole continuum. As the flow momentum increases, the two rotonlike spectra lean toward the left side. At a certain critical flow momentum, the energy minimum of the first collective mode hits zero, but this occurs before both the minimum of the second mode, and the lower boundary of the particle-hole continuum do.

## Appendix

The BS equation (51) written in the matrix form is  $(\hat{I} + U\hat{Z})\hat{\Psi} = 0$ , where  $\hat{I}$  is the  $16 \times 16$  unit matrix. The  $16 \times 16$  matrix  $\hat{Z}$  is defined as follows:

$$\hat{Z} = \begin{pmatrix} -\frac{K_{1414}}{2} & 0 & 0 & K_{1411} & 0 & \frac{K_{1111}}{2} & 0 & 0 & 0 & 0 & \frac{K_{1414}}{2} & 0 & K_{1114} & 0 & 0 & -\frac{K_{1111}}{2} \\ 0 & -K_{1111} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_{1411} & 0 \\ 0 & K_{1411} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -K_{4414} & 0 \\ -\frac{K_{4414}}{2} & 0 & 0 & K_{4411} & 0 & \frac{K_{1411}}{2} & 0 & 0 & 0 & 0 & \frac{K_{4414}}{2} & 0 & K_{1414} & 0 & 0 & -\frac{K_{1411}}{2} \\ 0 & 0 & 0 & 0 & -K_{1111} & 0 & 0 & 0 & 0 & 0 & 0 & K_{1414} & 0 & 0 & 0 & 0 \\ \frac{K_{1111}}{2} & 0 & 0 & 0 & 0 & -\frac{K_{1414}}{2} & -K_{1411} & 0 & 0 & -K_{1114} & -\frac{K_{1111}}{2} & 0 & 0 & 0 & 0 & \frac{K_{1414}}{2} \\ -\frac{K_{1411}}{2} & 0 & 0 & 0 & 0 & \frac{K_{4414}}{2} & K_{4411} & 0 & 0 & K_{1414} & \frac{K_{1411}}{2} & 0 & 0 & 0 & 0 & -\frac{K_{4414}}{2} \\ 0 & 0 & 0 & 0 & -K_{1411} & 0 & 0 & 0 & 0 & 0 & 0 & K_{4414} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{1114} & 0 & 0 & 0 & 0 & 0 & 0 & -K_{1444} & 0 & 0 & 0 & 0 \\ -\frac{K_{1114}}{2} & 0 & 0 & 0 & 0 & \frac{K_{1444}}{2} & K_{1414} & 0 & 0 & K_{1144} & \frac{K_{1114}}{2} & 0 & 0 & 0 & 0 & -\frac{K_{1444}}{2} \\ \frac{K_{1414}}{2} & 0 & 0 & 0 & 0 & -\frac{K_{4444}}{2} & -K_{4414} & 0 & 0 & -K_{1444} & -\frac{K_{1414}}{2} & 0 & 0 & 0 & 0 & \frac{K_{4444}}{2} \\ 0 & 0 & 0 & 0 & K_{1414} & 0 & 0 & 0 & 0 & 0 & 0 & -K_{4444} & 0 & 0 & 0 & 0 \\ -\frac{K_{1444}}{2} & 0 & 0 & K_{1414} & 0 & \frac{K_{1114}}{2} & 0 & 0 & 0 & 0 & \frac{K_{1444}}{2} & 0 & K_{1144} & 0 & 0 & -\frac{K_{1114}}{2} \\ 0 & -K_{1114} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_{1444} & 0 \\ 0 & K_{1414} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -K_{4444} & 0 \\ -\frac{K_{4444}}{2} & 0 & 0 & K_{4414} & 0 & \frac{K_{1414}}{2} & 0 & 0 & 0 & 0 & \frac{K_{4444}}{2} & 0 & K_{1444} & 0 & 0 & -\frac{K_{1414}}{2} \end{pmatrix}$$

The transposed matrix of  $\hat{\Psi}$  is given by:

$$\hat{\Psi}^T = \left( \Psi_{1,1}^Q \ \Psi_{1,2}^Q \ \Psi_{1,3}^Q \ \Psi_{1,4}^Q \ \Psi_{2,1}^Q \ \Psi_{2,2}^Q \ \Psi_{2,3}^Q \ \Psi_{2,4}^Q \ \Psi_{3,1}^Q \ \Psi_{3,2}^Q \ \Psi_{3,3}^Q \ \Psi_{3,4}^Q \ \Psi_{4,1}^Q \ \Psi_{4,2}^Q \ \Psi_{4,3}^Q \ \Psi_{4,4}^Q \right).$$

At zero temperature the elements of matrix  $\hat{Z}$  are:

$$\begin{aligned} K_{1111} &= \frac{1}{2} (I_{m,m} + I_{\tilde{\gamma},\tilde{\gamma}} - 2J_{\tilde{\gamma},m}), & K_{4444} &= \frac{1}{2} (I_{m,m} + I_{\tilde{\gamma},\tilde{\gamma}} + 2J_{\tilde{\gamma},m}), & K_{1144} &= \frac{1}{2} (I_{l,l} + I_{\gamma,\gamma} + 2J_{\gamma,l}) \\ K_{4411} &= \frac{1}{2} (I_{l,l} + I_{\gamma,\gamma} - 2J_{\gamma,l}), & K_{1114} &= \frac{1}{2} (-I_{l,m} + I_{\gamma,\tilde{\gamma}} + J_{l,\tilde{\gamma}} - J_{\gamma,m}), & K_{4414} &= \frac{1}{2} (I_{l,m} - I_{\gamma,\tilde{\gamma}} + J_{l,\tilde{\gamma}} - J_{\gamma,m}) \\ K_{1414} &= \frac{1}{2} (I_{l,l} - I_{\gamma,\gamma}), & K_{1444} &= \frac{1}{2} (I_{l,m} + I_{\gamma,\tilde{\gamma}} + J_{l,\tilde{\gamma}} + J_{\gamma,m}), & K_{1411} &= \frac{1}{2} (-I_{l,m} - I_{\gamma,\tilde{\gamma}} + J_{l,\tilde{\gamma}} - J_{\gamma,m}), \end{aligned}$$

where the following symbols are used:

$$I_{a,b} = \frac{1}{N} \sum_{\mathbf{k}} \frac{a_{\mathbf{k},\mathbf{Q}} b_{\mathbf{k},\mathbf{Q}} \varepsilon(\mathbf{k}, \mathbf{Q})}{\omega^2 - \varepsilon^2(\mathbf{k}, \mathbf{Q})}, \quad J_{a,b} = \frac{1}{N} \sum_{\mathbf{k}} \frac{\omega^2 a_{\mathbf{k},\mathbf{Q}} b_{\mathbf{k},\mathbf{Q}}}{\omega - \varepsilon^2(\mathbf{k}, \mathbf{Q})}.$$

Here,  $a$  and  $b$  are one of the form factors (50).

The above relations can be easily extended to the case of a moving lattice. In the presence of quasimomentum  $\mathbf{p}$  the following definitions and relationships hold:

$$u_{\mathbf{p}}(\mathbf{k}) = \sqrt{\frac{1}{2} [1 + \chi(\mathbf{k}; \mathbf{p}) / E_{\mathbf{p}}(\mathbf{k})]}, \quad v_{\mathbf{p}}(\mathbf{k}) = \sqrt{\frac{1}{2} [1 - \chi(\mathbf{k}; \mathbf{p}) / E_{\mathbf{p}}(\mathbf{k})]}, \quad E_{\mathbf{p}}(\mathbf{k}) = \sqrt{\chi^2(\mathbf{k}; \mathbf{p}) + \Delta_{\mathbf{p}}^2},$$

$$\chi(\mathbf{k}; \mathbf{p}) = \frac{1}{2} [\xi(\mathbf{p} + \mathbf{k}) + \xi(\mathbf{k} - \mathbf{p})], \quad \eta(\mathbf{k}; \mathbf{p}) = \frac{1}{2} [\xi(\mathbf{p} + \mathbf{k}) - \xi(\mathbf{k} - \mathbf{p})], \quad E_{\pm}(\mathbf{k}; \mathbf{p}) = \eta(\mathbf{k}; \mathbf{p}) \pm E_{\mathbf{p}}(\mathbf{k}).$$

At zero temperature the gap equation for  $\Delta_{\mathbf{p}}$  and the particle number equation are:

$$f = 1 - \frac{1}{N} \sum_{\mathbf{k}} \frac{\chi(\mathbf{k}; \mathbf{p})}{E_{\mathbf{p}}(\mathbf{k})}, \quad 1 = \frac{|U|}{N} \sum_{\mathbf{k}} \frac{1}{2E_{\mathbf{p}}(\mathbf{k})}.$$

In the presence of quasimomentum  $\mathbf{p}$  the elements of the  $16 \times 16$  determinant  $\hat{Z}$  are the same as in the case of a stationary lattice, but the definitions of  $I_{a,b}$  and  $J_{a,b}$  are as follows:

$$I_{a,b} = \frac{1}{N} \sum_{\mathbf{k}} \frac{a_{\mathbf{k},\mathbf{Q}}^{\mathbf{p}} b_{\mathbf{k},\mathbf{Q}}^{\mathbf{p}} \varepsilon_{\mathbf{p}}(\mathbf{k}, \mathbf{Q})}{[\omega + \Omega_{\mathbf{p}}(\mathbf{k}, \mathbf{Q})]^2 - \varepsilon_{\mathbf{p}}^2(\mathbf{k}, \mathbf{Q})}, \quad J_{a,b} = \frac{1}{N} \sum_{\mathbf{k}} \frac{a_{\mathbf{k},\mathbf{Q}}^{\mathbf{p}} b_{\mathbf{k},\mathbf{Q}}^{\mathbf{p}} [\omega + \Omega_{\mathbf{p}}(\mathbf{k}, \mathbf{Q})]}{[\omega + \Omega_{\mathbf{p}}(\mathbf{k}, \mathbf{Q})]^2 - \varepsilon_{\mathbf{p}}^2(\mathbf{k}, \mathbf{Q})},$$

where  $\varepsilon_{\mathbf{p}}(\mathbf{k}, \mathbf{Q}) = E_{\mathbf{p}}(\mathbf{k} + \mathbf{Q}) + E_{\mathbf{p}}(\mathbf{k})$  and  $\Omega_{\mathbf{p}}(\mathbf{k}, \mathbf{Q}) = \eta_{\mathbf{p}}(\mathbf{k}) - \eta_{\mathbf{p}}(\mathbf{k} + \mathbf{Q})$ .  $a$  and  $b$  are one of the following form factors:

$$\gamma_{\mathbf{k}, \mathbf{Q}}^{\mathbf{p}} = u_{\mathbf{p}}(\mathbf{k})u_{\mathbf{p}}(\mathbf{k} + \mathbf{Q}) + v_{\mathbf{p}}(\mathbf{k})v_{\mathbf{p}}(\mathbf{k} + \mathbf{Q}), \quad l_{\mathbf{k}, \mathbf{Q}}^{\mathbf{p}} = u_{\mathbf{p}}(\mathbf{k})u_{\mathbf{p}}(\mathbf{k} + \mathbf{Q}) - v_{\mathbf{p}}(\mathbf{k})v_{\mathbf{p}}(\mathbf{k} + \mathbf{Q}),$$

$$\tilde{\gamma}_{\mathbf{p}, \mathbf{k}, \mathbf{Q}} = u_{\mathbf{p}}(\mathbf{k})v_{\mathbf{p}}(\mathbf{k} + \mathbf{Q}) - v_{\mathbf{p}}(\mathbf{k})u_{\mathbf{p}}(\mathbf{k} + \mathbf{Q}), \quad m_{\mathbf{k}, \mathbf{Q}}^{\mathbf{p}} = u_{\mathbf{p}}(\mathbf{k})v_{\mathbf{p}}(\mathbf{k} + \mathbf{Q}) + v_{\mathbf{p}}(\mathbf{k})u_{\mathbf{p}}(\mathbf{k} + \mathbf{Q}).$$

It is well-known that the singularity of the integrals  $I_{a,b}$  and  $J_{a,b}$  at energies  $\omega = \varepsilon_{\mathbf{p}}(\mathbf{k}, \mathbf{Q}) - \Omega_{\mathbf{p}}(\mathbf{k}, \mathbf{Q})$  corresponds physically to the possibility of depairing into two fermion excitations with energies  $E_+(\mathbf{k} + \mathbf{Q}; \mathbf{p})$  and  $-E_-(\mathbf{k}; \mathbf{p})$ . The spectrum for this kind of excitation is known as

the particle-hole continuum.<sup>12</sup> The lower boundary of the particle-hole continuum is defined by the condition  $\min_{\mathbf{k}} (E_+(\mathbf{k} + \mathbf{Q}; \mathbf{p}) - E_-(\mathbf{k}; \mathbf{p}))$  where the minimum is to be taken over all the possible values of  $\mathbf{k}$ .

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<sup>1</sup> R. Laughlin, cond-mat/0209269.

<sup>2</sup> F. C. Zhang, Phys. Rev. Lett. **90**, 207002 (2003), J. Y. Gan, F. C. Zhang, and Z. B. Su, Phys. Rev. B **71**, 014508 (2005).

<sup>3</sup> M. Greiner, C. A. Regal, D. S. Jin, Nature **426** 537 (2003); S. Jochim, M. Bartenstein, A. Altmeyer, G. Hendl, S. Riedl, C. Chin, J. Hecker Denschlag, R. Grimm, Science **302** 2101 (2003); M. W. Zwierlein, C. A. Stan, C. H. Schunck, S. M. F. Raupach, S. Gupta, Z. Hadzibabic, W. Ketterle, Phys. Rev. Lett. **91** 250401 (2003); T. Bourdel, L. Khaykovich, J. Cubizolles, J. Zhang, F. Chevy, M. Teichmann, L. Tarruell, S. J. J. M. F. Kokkelmans, C. Salomon, Phys. Rev. Lett. **93** 050401 (2004).

<sup>4</sup> R. Ganesh, A. Paramekanti, and A. A. Burkov, Phys. Rev. A **80**, 043612 (2009).

<sup>5</sup> Y. Yunomae, D. Yamamoto, I. Danshita, N. Yokoshi, and S. Tsuchiya, Phys. Rev. A **80**, 063627 (2009).

<sup>6</sup> R. Coté and A. Griffin, Phys. Rev. B **37**, 4539 (1988).

<sup>7</sup> H. Chu and Y. C. Chang, Phys. Rev. B **54**, 5020 (1996).

<sup>8</sup> Z. Koinov, Phys. Rev. B **72**, 085203 (2005).

<sup>9</sup> R. Coté and A. Griffin, Phys. Rev. B, **48**, 10404 (1993).

<sup>10</sup> Z. G. Koinov, Physica Status Solidi (B), **247**, 140 (2010); Physica C **407**, 470 (2010).

<sup>11</sup> Z. G. Koinov, Ann. Phys. (Berlin) **522**, 693 (2010).

<sup>12</sup> R. Combescot, M. Yu. Kagan, and S. Stringari, Phys. Rev. A **74**, 042717 (2006).

<sup>13</sup> P. W. Anderson, Phys. Rev. **112**, 1900 (1958).

<sup>14</sup> G. Rickayzen, Phys. Rev. **115**, 795 (1959).

<sup>15</sup> L. Belkhir and M. Randeria, Phys. Rev. B **49**, 6829 (1994).

<sup>16</sup> G. Baym and L. P. Kadanoff, Phys. Rev., **124**, 287 (1961).

<sup>17</sup> G. Baym, Phys. Rev. **127**, 1391 (1962).

<sup>18</sup> C. Itzykson and J. Zuber, Quantum Field Theory, McGraw-Hill, NY 1980.

<sup>19</sup> C. De Dominicis and P. Martin, J. Math. Phys., **5**, 430 (1964).

<sup>20</sup> W.F. Brinkman, J.W. Serene, and P.W. Anderson, Phys. Rev. A **10**, 2386 (1974).